

HOMOGENEOUS ROTA-BAXTER OPERATORS ON 3-LIE ALGEBRA A_ω

RUIPU BAI AND YINGHUA ZHANG

ABSTRACT. In the paper we study homogeneous Rota-Baxter operators with weight zero on the infinite dimensional simple 3-Lie algebra A_ω over a field F ($chF = 0$) which is realized by an associative commutative algebra A and a derivation Δ and an involution ω (Lemma 2.4). A homogeneous Rota-Baxter operator on A_ω is a linear map R of A_ω satisfying $R(L_m) = f(m)L_m$ for all generators of A_ω , where $f : A_\omega \rightarrow F$. We proved that R is a homogeneous Rota-Baxter operator on A_ω if and only if R is the one of the five possibilities $R_{0_1}, R_{0_2}, R_{0_3}, R_{0_4}$ and R_{0_5} , which are described in Theorem 3.2, 3.12, 3.15, 3.19 and 3.21. By the five homogeneous Rota-Baxter operators R_{0_i} , we construct new 3-Lie algebras $(A, [, ,]_i)$ for $1 \leq i \leq 5$, such that R_{0_i} is the homogeneous Rota-Baxter operator on 3-Lie algebra $(A, [, ,]_i)$, respectively.

1. INTRODUCTION

Rota-Baxter operators were originally defined on associative algebras by G. Baxter to solve an analytic formula in probability [12] and populated by the work of Cartier and Rota [13, 35, 36]. They have been closely related to many fields in mathematics and mathematical physics. Rota-Baxter algebras have played an important role in the Hopf algebra approach of renormalization of perturbative quantum field theory of Connes and Kreimer [14, 16, 17], as well as in the application of the renormalization method in solving divergent problems in number theory [23, 29].

Rota-Baxter operators on a Lie algebra are an operator form of the classical Yang-Baxter equations and contribute to the study of integrable systems [4, 6, 7]. Semenov-Tian-Shanskys fundamental work [37] shows that a Rota-Baxter operator of weight 0 on a Lie algebra is exactly the operator form of the classical Yang-Baxter equation (CYBE), which was regarded as a classical limit of the quantum Yang-Baxter equation [34]. Whereas the latter is also an important topic in many fields such as symplectic geometry, integrable systems, quantum groups and quantum field theory [1, 5, 16, 19, 20, 21, 22, 28, 23, 35, 36].

Rota-Baxter n -algebras and differential n -algebras were first introduced in [39], they are the generalization of Rota-Baxter algebras to the multiple algebraic systems. We know that n -Lie algebras [18] are a type of multiple algebraic systems appearing in many fields of mathematics and mathematical physics [30, 38, 3, 26, 27, 25, 24, 31]. Especially, 3-Lie algebras and metric 3-Lie algebras are applied to the study of the supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes; the Bagger-Lambert theory has a novel local gauge symmetry which is based on a metric 3-Lie algebra; the n -Jacobi identity in n -Lie algebras can be regarded as a generalized Plucker relation in the physics literature. The theory of n -Lie algebras has been widely studied [32, 33, 10, 11, 8, 9, 2]. For the recent years, the most interesting work on the structure of n -Lie algebras is the realization of n -Lie

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algebras ($n \geq 3$) from well know algebras, for example, from Lie algebras, associative algebras, commutative associative algebras, cubic matrices, etc. [8, 40, 41, 15, 42, 43].

Authors in paper [39] provided the Rota-Baxter operator on n -Lie algebras and studied the structure of Rota-Baxter 3-Lie algebras, and also gave the method to realize Rota-Baxter 3-Lie algebras from Rota-Baxter 3-Lie algebras, Rota-Baxter Lie algebras, Rota-Baxter pre-Lie algebras and Rota-Baxter commutative associative algebras and derivations.

In this paper we investigate a class of Rota-Baxter operators with weight zero on the simple 3-Lie algebra A_ω , which is constructed from a commutative associative algebra A and a derivation Δ and an involution ω which satisfies $\Delta\omega + \omega\Delta = 0$ [40]. This on one hand further studies the structures of the simple Rota-Baxter 3-Lie algebra, and on the other hand provides a rich source of examples for Rota-Baxter 3-Lie algebras.

The article is organized as follows. Section 2 describes concepts of Rota-Baxter operators with weights for general n -ary algebras and some results which are used in the paper. In Section 3 is devoted to the homogeneous Rota-Baxter operators on A_ω with weight zero. At last of the paper, new 3-Lie algebras are constructed by the homogeneous Rota-Baxter operators on A_ω .

In this paper, we suppose that F is a field of characteristic zero, and Z is the set of integer numbers.

2. PRELIMINARY

An **n -Lie algebra** [18] is a vector space A over a field F endowed with an n -ary multi-linear skew-symmetric operation $[x_1, \dots, x_n]$ satisfying the n -Jacobi identity

$$(1) \quad [[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^n [x_1, \dots, [x_i, y_2, \dots, y_n], \dots, x_n].$$

In particular, a **3-Lie algebra** is a vector space A endowed with a ternary multi-linear skew-symmetric operation satisfying for all $x_1, x_2, x_3, y_2, y_3 \in A$.

$$(2) \quad [[x_1, x_2, x_3], y_2, y_3] = [[x_1, y_2, y_3], x_2, x_3] + [[x_2, y_2, y_3], x_3, x_1] + [[x_3, y_2, y_3], x_1, x_2].$$

Definition 2.1. Let $\lambda \in F$ be fixed.

- (a) An **n -(nonassociative) algebra** over a field F is a pair $(A, \langle \cdot, \dots, \cdot \rangle)$ consisting of a vector space A over F and a multilinear multiplication

$$\langle \cdot, \dots, \cdot \rangle : A^{\otimes n} \rightarrow A.$$

- (b) A **derivation of weight λ** on an n -algebra $(A, \langle \cdot, \dots, \cdot \rangle)$ is a linear map $d : A \rightarrow A$ such that,

$$(3) \quad d(\langle x_1, \dots, x_n \rangle) = \sum_{\emptyset \neq I \subseteq [n]} \lambda^{|I|-1} \langle \check{d}(x_1), \dots, \check{d}(x_i), \dots, \check{d}(x_n) \rangle,$$

where $\check{d}(x_i) := \check{d}_I(x_i) := \begin{cases} d(x_i), & i \in I, \\ x_i, & i \notin I \end{cases}$ for all $x_1, \dots, x_n \in A$. Then A is called a

differential n -algebra of weight λ . In particular, a **differential 3-algebra of weight λ** is a 3-algebra $(A, \langle \cdot, \cdot, \cdot \rangle)$ with a linear map $d : A \rightarrow A$ such that

$$(4) \quad \begin{aligned} d(\langle x_1, x_2, x_3 \rangle) &= \langle d(x_1), x_2, x_3 \rangle + \langle x_1, d(x_2), x_3 \rangle + \langle x_1, x_2, d(x_3) \rangle \\ &+ \lambda \langle d(x_1), d(x_2), x_3 \rangle + \lambda \langle d(x_1), x_2, d(x_3) \rangle + \lambda \langle x_1, d(x_2), d(x_3) \rangle \\ &+ \lambda^2 \langle d(x_1), d(x_2), d(x_3) \rangle. \end{aligned}$$

(c) A **Rota-Baxter operator of weight λ** on $(A, \langle, \cdots, \rangle)$ is a linear map $R : A \rightarrow A$ such that

$$(5) \quad \langle R(x_1), \cdots, R(x_n) \rangle = R \left(\sum_{\emptyset \neq I \subseteq [n]} \lambda^{|I|-1} \langle \hat{R}(x_1), \cdots, \hat{R}(x_i), \cdots, \hat{R}(x_n) \rangle \right),$$

where $\hat{R}(x_i) := \hat{R}_I(x_i) := \begin{cases} x_i, & i \in I, \\ R(x_i), & i \notin I \end{cases}$ for all $x_1, \cdots, x_n \in A$. Then A is called a **Rota-Baxter n -algebra of weight λ** . In particular, a **Rota-Baxter 3-algebra** is a 3-algebra (A, \langle, \rangle) with a linear map $P : A \rightarrow A$ such that

$$(6) \quad \begin{aligned} \langle R(x_1), R(x_2), R(x_3) \rangle &= R(\langle R(x_1), R(x_2), x_3 \rangle + \langle R(x_1), x_2, R(x_3) \rangle + \langle x_1, R(x_2), R(x_3) \rangle \\ &\quad + \lambda \langle R(x_1), x_2, x_3 \rangle + \lambda \langle x_1, R(x_2), x_3 \rangle + \lambda \langle x_1, x_2, R(x_3) \rangle \\ &\quad + \lambda^2 \langle x_1, x_2, x_3 \rangle). \end{aligned}$$

Lemma 2.2. [39] Let $(A, \langle, \cdots, \rangle)$ be an n -algebra over F . An invertible linear mapping $P : A \rightarrow A$ is a Rota-Baxter operator of weight λ on A if and only if P^{-1} is a differential operator of weight λ on A .

Lemma 2.3. Let $(A, \langle, \cdots, \rangle, R)$ be a Rota-Baxter n -algebra over F with weight 0. Then for all $\lambda \in F$, $\lambda \neq 0$, $(A, \langle, \cdots, \rangle, \lambda R)$ is a Rota-Baxter n -algebra with weight 0.

Proof. The result follows from Eq. (5), directly. □

Lemma 2.4. [40] Let A be a vector space with a basis $\{L_n \mid n \in \mathbb{Z}\}$ over field F . Then A is a simple 3-Lie algebra in the multiplication

$$(7) \quad [L_l, L_m, L_n] = \begin{vmatrix} (-1)^l & (-1)^m & (-1)^n \\ 1 & 1 & 1 \\ l & m & n \end{vmatrix} L_{l+m+n-1}, \text{ for all } l, m, n \in \mathbb{Z}.$$

In the following, the 3-Lie algebra A in Lemma 2.4 is denoted by \mathbf{A}_ω , and the determinant

$$\begin{vmatrix} (-1)^l & (-1)^m & (-1)^n \\ 1 & 1 & 1 \\ l & m & n \end{vmatrix} \text{ is denoted by } D(l, m, n).$$

Lemma 2.5. The determinant $D(l, m, n) = 0$ if and only if

$(l - m)(l - n)(m - n) = 0$, or $l = 2k + 1, m = 2s + 1, n = 2t + 1$, or $l = 2k, m = 2s, n = 2t$, for all $k, s, t \in \mathbb{Z}$.

Proof. The result follows from a direct computation. □

3. HOMOGENEOUS ROTA-BAXTER OPERATORS WITH WEIGHT 0 ON 3-LIE ALGEBRA A_ω

In this section we discuss Rota-Baxter operators with weight 0 on the 3-Lie algebra A_ω .

By Definition 2.1, if $(A, [,], R)$ is a Rota-Baxter 3-Lie algebra of weight $\lambda = 0$. Then the linear map $R : A \rightarrow A$ satisfies that for all $x_1, x_2, x_3 \in A$,

$$(8) \quad [R(x_1), R(x_2), R(x_3)] = R([R(x_1), R(x_2), x_3] + [R(x_1), x_2, R(x_3)] + [x_1, R(x_2), R(x_3)]).$$

A **homogeneous Rota-Baxter operator** R on the 3-Lie algebra A_ω is a Rota-Baxter operator satisfies that there exists $f : Z \rightarrow F$ satisfying

$$(9) \quad R(L_m) = f(m)L_m, \quad \forall m \in Z.$$

Theorem 3.1. *Let $R : A_\omega \rightarrow A_\omega$ be a linear map defined as Eq. (9). Then R is a homogeneous Rota-Baxter operator of weight 0 on A_ω if and only if f satisfies for all $l, m, n \in Z$,*

$$(10) \quad f(l)f(m)f(n)D(l, m, n) = (f(l)f(n) + f(m)f(n) + f(l)f(m))f(l + m + n - 1)D(l, m, n).$$

Proof. By Eqs. (7), (8) and (9), we have

$$[R(L_l), R(L_m), R(L_n)] = f(l)f(m)f(n)D(l, m, n)L_{l+m+n-1},$$

and

$$\begin{aligned} & R([R(L_l), R(L_m), L_n] + [R(L_l), L_m, R(L_n)] + [L_l, R(L_m), R(L_n)]) = \\ & (f(l)f(m) + f(l)f(n) + f(m)f(n))f(l + m + n - 1)D(l, m, n)L_{l+m+n-1}. \end{aligned}$$

Therefore, R is a homogeneous Rota-Baxter operator on A_ω if and only if Eq. (10) holds. \square

3.1. Homogeneous Rota-Baxter operators with $f(0) + f(1) \neq 0$. In this section we discuss the homogeneous Rota-Baxter operators with weight 0 defined by Eq. (9) with $f(0) + f(1) \neq 0$.

Theorem 3.2. *Let $R : A_\omega \rightarrow A_\omega$ be a linear map defined as Eq. (9) with $f(0) + f(1) \neq 0$. Then R is a homogeneous Rota-Baxter operator on A_ω if and only if*

$$f(m) = 0, \text{ for all } m \in Z, \text{ and } m \neq 0, 1.$$

Proof. If f satisfies $f(m) = 0$, for all $m \in Z$, and $m \neq 0, 1$. By a direct computation R is a homogeneous Rota-Baxter operator.

Conversely, if R is a homogeneous Rota-Baxter operator with $f(0) + f(1) \neq 0$. Then Eq. (8) of the case $l = 0, n = 1$ becomes

$$f(0)f(m)f(1) = \{f(0)f(1) + f(m)f(1) + f(0)f(m)\}f(m), \forall m \in Z, m \neq 0, 1.$$

Since $f(0) + f(1) \neq 0$, we have $f(m)^2 = 0$, for all $m \in Z$ and $m \neq 0, 1$. The proof is completed. \square

3.2. Homogeneous Rota-Baxter operators with $f(0) + f(1) = 0$.

Lemma 3.3. *Let $R : A_\omega \rightarrow A_\omega$ be a linear map of A defined as Eq. (9) with $f(0) + f(1) = 0$. Then R is a homogeneous Rota-Baxter operator if and only if for all $l, m, n \in Z$,*

$$(11) \quad \begin{aligned} f(2l+1)f(2m+1)f(2n) &= (f(2l+1)f(2m+1) + f(2l+1)f(2n) \\ &\quad + f(2m+1)f(2n))f(2l+2m+2n+1), \quad m \neq l, \end{aligned}$$

$$(12) \quad \begin{aligned} f(2l+1)f(2m)f(2n) &= (f(2l+1)f(2m) + f(2l+1)f(2n) \\ &\quad + f(2m)f(2n))f(2l+2m+2n), \quad m \neq n. \end{aligned}$$

Proof. For all $l, m, n \in \mathbb{Z}$ and $l \neq m$ and $m \neq n$, the determinant $D(2l+1, 2m+1, 2n) \neq 0$ and $D(2l+1, 2m, 2n) \neq 0$. Thanks to Eq. (7) and Eq. (9), we obtain Eq. (11) and Eq. (12). \square

3.2.1 Homogeneous Rota-Baxter operators with $f(0) = -f(1) \neq 0$.

Now we discuss the case $f(0) + f(1) = 0$, but $f(0) \neq 0$. By Lemma 2.3, we can suppose $f(0) = 1$, then $f(1) = -1$.

Corollary 3.4. *Let R be a homogeneous Rota-Baxter operator with $f(0) = -f(1) = 1$. Then we have for all $l, m, n \in \mathbb{Z}$,*

- 1) $f(2l+1)f(2m+1) = (f(2l+1) + f(2m+1) + f(2l+1)f(2m+1))f(2l+2m+1), l \neq m,$
- 2) $f(2l+1)f(2m) = (f(2l+1) + f(2m) + f(2l+1)f(2m))f(2l+2m), m \neq 0,$
- 3) $f(2l+1)f(2n) = (f(2l+1) + f(2n) - f(2l+1)f(2n))f(2l+2n+1), l \neq 0,$
- 4) $f(2m)f(2n) = (f(2m) + f(2n) - f(2m)f(2n))f(2m+2n), m \neq n.$

Proof. The result follows from Lemma 3.3 and $f(0) = -f(-1) = 1$. \square

Theorem 3.5. *Let R be a homogeneous Rota-Baxter operator with $f(0) = -f(1) = 1$. Then we have*

$$(13) \quad f(1-m) + f(m) = 0, \text{ for all } m \in \mathbb{Z}.$$

Proof. According to Corollary 3.4, for all $n \in \mathbb{Z}$ and $n \neq 0$, we have

$$f(2m+1)(f(2m+2n) - f(2m+2n+1)) + f(2n)(f(2m+2n) - f(2m+2n+1)) \\ + f(2m+1)f(2n)(f(2m+2n) + f(2m+2n+1)) = 0.$$

Then in the case $m = -n \neq 0$, we have $f(2m+1) + f(-2m) = 0$, and

$$f(2m+1) + f(1-(2m+1)) = 0.$$

Similarly, we have $f(1-2(-m)) + f(2(-m)) = 0$, for all $m \in \mathbb{Z}$. It follows Eq. (13). \square

Corollary 3.6. *If R is a homogeneous Rota-Baxter operator on A_ω satisfying that $f(0) = -f(1) = 1$, and there exist $k, l, m, n \in \mathbb{Z}$ such that $f(2k) \neq 0, f(2l) \neq 0, f(2m+1) \neq 0, f(2n+1) \neq 0$, where the product $(k-l)(m-n)klmn \neq 0$. Then we have*

- 1) $f(2k+2l) \neq 0,$ 2) $f(2k+2m) \neq 0,$ 3) $f(2k+2m+1) \neq 0,$
- 4) $f(2m+2n+1) \neq 0,$ 5) $f(1-2k+2m) \neq 0, k \neq -m,$
- 6) $f(4k) \neq 0,$ 7) $f(2m+2n+2k+1) \neq 0,$
- 8) $f(2m+2k+2l) \neq 0,$ 9) $f(2k-2m) \neq 0, k \neq -m,$
- 10) $f(1-2k-2m) \neq 0,$ 11) $f(1-4k) \neq 0.$

Proof. The results 1), 2), 3), 4), 5) and 6) follow from Corollary 3.4 and $f(0) = -f(1) = 1$, the results 7) and 8) follow from Lemma 3.3, and the results 9), 10) and 11) follow from Theorem 3.5. \square

Lemma 3.7. *Let $R : A_\omega \longrightarrow A_\omega$ be a linear map defined by Eq. (9). If f satisfies $f(0) = -f(1) = 1$ and that there exist finite distinct integers m_i such that $f(m_i) \neq 0$ and $f(m) = 0$ for $m \in \mathbb{Z}$*

and $m \neq m_i$, then R is not a homogeneous Rota-Baxter operator on A_ω , where $1 \leq i \leq t$, and $m, m_i \neq 0, 1$.

Proof. If R is a homogeneous Rota-Baxter operator. Thanks to Theorem 3.5, $f(1 - m_i) = -f(m_i) \neq 0$ for $1 \leq i \leq t$. We obtain $t \geq 2$. Without loss of generality, suppose m_1 is odd, then $m_2 = 1 - m_1$ is even and $f(m_2) \neq 0$. Thanks to the result 6) in Corollary 3.6, $f(2nm_2) \neq 0$ for all $n \in \mathbb{Z}$. It contradicts $t < \infty$. It follows the result. \square

Lemma 3.8. *Let $R : A_\omega \rightarrow A_\omega$ be a linear map defined by Eq.(9). If f satisfies $f(0) = -f(1) = 1$ and that there exist finite distinct integers m_i such that $f(m_i) = 0$ and $f(m) \neq 0$ for $m \in \mathbb{Z}$ and $m \neq m_i$, then R is not a homogeneous Rota-Baxter operator on A_ω , where $1 \leq i \leq t$, and $m_i \neq 0, 1$.*

Proof. If R is a homogeneous Rota-Baxter operator. Thanks to Theorem 3.5, $f(1 - m_i) = -f(m_i) = 0$ for $1 \leq i \leq t$, and for all $n \neq m_i$, $f(n) = -f(1 - n) \neq 0$. It follows that there exist infinite odd $2l + 1 \in \mathbb{Z}$ such that $f(2l + 1) \neq 0$. Without loss of generality, suppose m_1 is even, then by Corollary 3.4, there exist infinite odd $2l + 1 \in \mathbb{Z}$, such that $f(2l + 2m_1) = 0$. It contradicts to $t < \infty$. \square

Theorem 3.9. *Let $R : A_\omega \rightarrow A_\omega$ be a linear map defined by Eq.(9). If R is a homogeneous Rota-Baxter operator on A_ω with $f(0) = -f(1) = 1$ and that there exists $m \neq 0, 1$ such that $f(m) \neq 0$. Then there exists a positive integer m_0 such that for $m \in \mathbb{Z}$, $f(m) \neq 0$ if and only if $m \in W = \{2m_0k | k \in \mathbb{Z}\} \cup \{1 - 2m_0k | k \in \mathbb{Z}\}$.*

Proof. From Theorem 3.5, there exists $W = \{2x_k | k \in \mathbb{Z}\} \cup \{1 - 2x_k | k \in \mathbb{Z}\} \subset \mathbb{Z}$ satisfying that $f(m) \neq 0$ if and only if $m \in W$. Thanks to Lemma 3.7 and 3.8, W is an infinite subset of \mathbb{Z} . From Corollary 3.6, we can suppose that for all $k, s \in \mathbb{Z}$, $2x_k < 2x_s$ if and only if $k < s$, and $x_{-1} < 0 < x_1$.

By the result 2) of Corollary 3.6 and $2x_2 \in W, -2x_1 + 1 \in W$, we have $2(x_2 - x_1) \in W$. Thanks to $0 < x_2 - x_1 < x_2, x_2 = 2x_1$.

Now suppose that $x_k - x_{k-1} = x_1$, for $k > 0$, that is, $x_k = kx_1$. Since $2x_{k+1} \in W, 2x_{k-1} \in W, -2x_k + 1 \in W$, by the result 8) in Corollary 3.6, we have $2(x_{k+1} - x_k + x_{k-1}) = 2(x_{k+1} - x_1) \in W$. Thanks to $x_{k+1} - x_1 < x_{k+1}$ and $x_{k-1} = x_k - x_1 < x_{k+1} - x_1, x_{k-1} < x_{k+1} - x_1 < x_{k+1}$. Therefore, $x_{k+1} - x_1 = x_k$, that is, $x_{k+1} = x_k + x_1 = (k + 1)x_1$.

By the completely similar discussion, we have that for all $k < 0$, $x_k = -kx_{-1}$.

Since $2x_{-1} \in W, 2x_1 \in W$, from the result 1) in Corollary 3.6, we have that $2(x_{-1} + x_1) \in W$. From $x_{-1} < 0 < x_1$, and $x_{-1} < x_{-1} + x_1 < x_1$, we have $x_{-1} + x_1 = 0$, that is, $x_{-1} = -x_1$. Denote $m_0 = x_1$. Then for all $2x_k \in W, 2x_k = 2m_0k$. \square

For positive integer m_0 , denote

$$W_{m_0} = \{2m_0k | k \in \mathbb{Z}\} \cup \{1 - 2m_0k | k \in \mathbb{Z}\}.$$

If f satisfies that $f(m) \neq 0$ if and only if $m \in W_{m_0}$, then W_{m_0} is called an m_0 -supporter of R .

Corollary 3.10. *Let R be a Homogeneous Rota-Baxter operator with $f(0) = -f(1) = 1$. If there exist integer k such that $f(2k) \neq 0$, then we have $f(-2k) \neq 0$ and $f(1 + 2k) \neq 0$.*

Proof. The result follows from Theorem 3.9 and Theorem 3.5, directly. \square

Lemma 3.11. *Let R be a homogeneous Rota-Baxter operator with $f(0) = -f(1) = 1$, and W_{m_0} be its m_0 -supporter. Then we have $f(2m_0) \neq \frac{1}{2}$, and for all $k, k_1, k_2, k_3 \in \mathbb{Z}$,*

$$(14) \quad \frac{1}{2f(2m_0k)} + \frac{1}{2f(-2m_0k)} = 1, \quad \frac{1}{2f(2m_0k)} - \frac{1}{2f(1+2m_0k)} = 1,$$

$$(15) \quad \frac{1}{f(2m_0k_2)} + \frac{1}{f(2m_0k_3)} = \frac{1}{f(2m_0k_1)} + \frac{1}{f(-2m_0k_1 + 2m_0k_2 + 2m_0k_3)}, \quad k_2 \neq k_3.$$

Proof. By the result 4) in Corollary 3.4, for all $k \in \mathbb{Z}$ and $k \neq 0$, we have

$$f(2m_0k)f(-2m_0k) = f(2m_0k) + f(-2m_0k) - f(2m_0k)f(-2m_0k).$$

It follows Eq. (14), and $f(2m_0) \neq \frac{1}{2}$.

According to Lemma 3.3 and Theorem 3.5, for all $m, n \in \mathbb{Z}$ and $m \neq n$, we have

$$-f(2l)f(2m)f(2n) = \{-f(2l)f(2m) - f(2l)f(2n) + f(2m)f(2n)\}f(-2l + 2m + 2n).$$

Then in the case $l = m_0k_1, m = m_0k_2, n = m_0k_3$, we obtain Eq. (15). \square

Theorem 3.12. *Let $R : A_\omega \rightarrow A_\omega$ be a linear map defined as Eq. (9) with $f(0) = -f(1) = 1$. Then R is a homogeneous Rota-Baxter operator on A_ω if and only if $f(m) = 0$ for all $m \in \mathbb{Z}$, $m \neq 0, 1$; or there exists a positive integer m_0 and an element $a \in F$, $a \neq \frac{k-1}{k}$ for $k \in \mathbb{Z} - \{0\}$, such that W_{m_0} is an m_0 -supporter of R and*

$$(16) \quad f(2m_0k) = -f(1 - 2m_0k) = \frac{1}{ka - (k-1)}, \quad \forall k \in \mathbb{Z}.$$

Further, in the case $m_0 = 1$, R is an invertible Rota-Baxter operator on A_ω , therefore, R^{-1} is an invertible derivation of A_ω , and

$$R^{-1}(L_{2k}) = (ka - (k-1))L_{2k}, \quad R^{-1}(L_{1-2k}) = (-ka + (k-1))L_{1-2k}, \quad \forall k \in \mathbb{Z}.$$

Proof. If R is a homogeneous Rota-Baxter operator on A_ω and there exists $m \neq 0, 1$ such that $f(m) \neq 0$, then by Theorem 3.9, there exists a positive integer m_0 such that W_{m_0} is an m_0 -supporter of R . Suppose $f(2m_0) = \frac{1}{a}$, then by Lemma 3.11 $a \neq 2$.

Now suppose that for positive integer k satisfies $f(2m_0k) = \frac{1}{ka - (k-1)}$. By Lemma 3.11,

$$\frac{1}{f(2m_0(k+1))} + 1 = \frac{1}{f(2m_0k)} + \frac{1}{f(2m_0)} = ka - (k-1) + a,$$

that is, $f(2m_0(k+1)) = \frac{1}{(k+1)a - k}$, and $a \neq \frac{k-1}{k}$.

Since

$$\frac{1}{f(2m_0)} + \frac{1}{f(-2m_0)} = 2,$$

we have $f(-2m_0) = \frac{1}{2-a} = \frac{1}{-a-(-1-1)}$. Now suppose that for negative integer k , $f(2m_0k) = \frac{1}{ka-(k-1)}$. From

$$\frac{1}{f(2m_0(k-1))} + 1 = \frac{1}{f(2m_0k)} + \frac{1}{f(-2m_0)} = ka - (k-1) + 2 - a,$$

we have

$$f(2m_0(k-1)) = \frac{1}{(k-1)a - (k-2)}, \quad \text{and} \quad a \neq \frac{k-2}{k-1}.$$

It follows Eq. (16).

Conversely, since for all $2l, 2m, 2n \notin W_{m_0}, l \neq m$,

$$f(\pm 2l) = f(\pm 2m) = f(\pm 2n) = 0, \quad f(1 \pm 2l) = f(1 \pm 2m) = f(1 \pm 2n) = 0,$$

the identity (10) holds. So we only need to prove that Eq. (10) holds for the following cases.

1) The case $2l, 2m \notin W_{m_0}, 2n \in W_{m_0}, l \neq m$. By Theorem 3.9 and Theorem 3.5

$f(\pm 2l) = f(\pm 2m) = f(1 \pm 2l) = f(1 \pm 2m) = 0, f(\pm 2m \pm 2n \pm 2l) = f(\pm 2m \pm 2n \pm 2l + 1) = 0$. Then Eq. (10) holds.

2) For the case $2l \notin W_{m_0}, 2m, 2n \in W_{m_0}$, and $m \neq n$. We have $f(\pm 2l) = f(1 \pm 2l) = 0, f(\pm 2l \pm 2m \pm 2n) = f(1 \pm 2l \pm 2m \pm 2n) = 0$. Then Eq. (10) holds.

3) For the case $2l, 2m, 2n \in W_{m_0}, l \neq m, l \neq n$ and $m \neq n$. Suppose $2l = 2m_0k_1, 2m = 2m_0k_2, 2n = 2m_0k_3 \in W_{m_0}$. From

$$\begin{aligned} f(1-2l)f(2m)f(2n) &= \frac{-1}{k_1a - (k_1-1)} \frac{1}{k_2a - (k_2-1)} \frac{1}{k_3a - (k_3-1)}, \\ (f(1-2l)f(2m) + f(1-2l)f(2n) + f(2m)f(2n))f(2(m+n-l)) \\ &= \left(\frac{-1}{k_1a - (k_1-1)} \frac{1}{k_2a - (k_2-1)} + \frac{-1}{k_1a - (k_1-1)} \frac{1}{k_3a - (k_3-1)} \right. \\ &\quad \left. + \frac{1}{k_2a - (k_2-1)} \frac{1}{k_3a - (k_3-1)} \right) \frac{1}{(-k_1+k_2+k_3)a - (-k_1+k_2+k_3-1)} \\ &= \frac{-1}{k_1a - (k_1-1)} \frac{1}{k_2a - (k_2-1)} \frac{1}{k_3a - (k_3-1)}, \\ f(1-2l)f(1-2m)f(2n) &= \frac{-1}{k_1a - (k_1-1)} \frac{-1}{k_2a - (k_2-1)} \frac{1}{k_3a - (k_3-1)}, \\ (f(1-2l)f(1-2m) + f(1-2l)f(2n) + f(1-2m)f(2n))f(1-2(l+m-n)) \\ &= \left(\frac{-1}{k_1a - (k_1-1)} \frac{-1}{k_2a - (k_2-1)} + \frac{-1}{k_1a - (k_1-1)} \frac{1}{k_3a - (k_3-1)} \right. \\ &\quad \left. + \frac{-1}{k_2a - (k_2-1)} \frac{1}{k_3a - (k_3-1)} \right) \frac{-1}{(k_1+k_2-k_3)a - (k_1+k_2-k_3-1)} \\ &= \frac{1}{k_1a - (k_1-1)} \frac{1}{k_2a - (k_2-1)} \frac{1}{k_3a - (k_3-1)}, \end{aligned}$$

identity (10) holds.

Summarizing above discussion, we obtain the result. \square

Let $m_0 = 1$, and $a = 3$. By Theorem 3.12, the linear map $R : A_\omega \rightarrow A_\omega$ defined by

$$R(L_{2k}) = \frac{1}{3k - (k-1)}L_{2k} = \frac{1}{2k+1}L_{2k}, \quad R(L_{1-2k}) = -\frac{1}{2k+1}L_{1-2k}, \quad k \in \mathbb{Z},$$

is a homogeneous Rota-Baxter operator on A_ω , and R is an invertible Rota-Baxter operator. Therefore, $D = R^{-1} : A_\omega \rightarrow A_\omega$ satisfying

$$D(L_{2k}) = (2k+1)L_{2k}, \quad D(L_{1-2k}) = -(2k+1)L_{1-2k}, \quad k \in \mathbb{Z},$$

is an invertible derivation of A_ω .

If $m_0 = 3$, and $a = \sqrt{2}$. Then the linear map $R : A_\omega \rightarrow A_\omega$ defined by

$$R(L_{6k}) = \frac{1}{\sqrt{2}k - (k-1)}L_{6k}, \quad R(L_{1-6k}) = -\frac{1}{\sqrt{2}k - (k-1)}L_{1-6k}, \quad k \in \mathbb{Z},$$

and others are zero, is a homogeneous Rota-Baxter operator on A_ω . But R is degenerate.

3.2.2 Homogeneous Rota-Baxter operators with $f(0) = f(1) = 0$

In this section we discuss the case $f(0) = f(1) = 0$.

Lemma 3.13. *Let $R : A_\omega \rightarrow A_\omega$ be a homogeneous Rota-Baxter operator on A_ω with $f(0) = f(1) = 0$. Then R satisfies that for all $l, m, n \in \mathbb{Z}$,*

- 1) $f(2l+1)f(2m+1)f(2l+2m+1) = 0, \quad l \neq m.$
- 2) $f(2m+1)f(2n)f(2m+2n+1) = 0, \quad m \neq 0.$
- 3) $f(2l+1)f(2m)f(2l+2m) = 0, \quad m \neq 0.$
- 4) $f(2m)f(2n)f(2m+2n) = 0, \quad m \neq n.$

Proof. The result follows from $D(2l+1, 2m+1, 0) \neq 0, D(1, 2m+1, 2n) \neq 0, D(2l+1, 2m, 0) \neq 0, D(1, 2m, 2n) \neq 0$, and Lemma 3.3. □

Corollary 3.14. *Let $R : A_\omega \rightarrow A_\omega$ be a homogeneous Rota-Baxter operator with $f(0) = f(1) = 0$, and there exist $k, l, m, n \in \mathbb{Z}$ such that $(k-l)(m-n)klmn \neq 0, f(2k) \neq 0, f(2l) \neq 0, f(2m+1) \neq 0, f(2n+1) \neq 0$. Then we have*

- 1) $f(2k+2l) = 0, \quad 2) f(2k+2m) = 0, \quad 3) f(2k+2m+1) = 0,$
- 4) $f(2m+2n+1) = 0, \quad 5) f(2m+2n+2k+1) \neq 0,$
- 6) $f(2m+2k+2l) \neq 0, \quad 7) f(2k-2m) = 0, \quad k \neq -m, \quad 8) f(4k) = 0.$

Proof. The result 1), 2), 3) and 4) follow from the result 4), 3), 2) and 1) in Lemma 3.13, respectively. The result 5) and 6) follow from Eq. (11) and Eq. (12), respectively. The result 7) and 8) follow from the result 4) and 3) in Lemma 3.13, respectively. □

Theorem 3.15. *Let $R : A_\omega \rightarrow A_\omega$ be a homogeneous Rota-Baxter operator with $f(0) = f(1) = 0$, and there exist $m_1, \dots, m_s \in \mathbb{Z}$ such that $f(m_i) \neq 0$ and $f(m) = 0$ for all $m \neq m_i$, where $m_i \neq 0, 1, 1 \leq i \leq s$. Then we have*

- 1) $s = 1$, and then we can suppose $f(m_1) = 1, f(m) = 0$ for all $m \in \mathbb{Z}, m \neq m_1.$
- 2) $s = 2$ and $m_1 + m_2 = 1$, so we can suppose that $f(m_1) = 1, f(1-m_1) = b$, and $f(m) = 0$ for all $m \in \mathbb{Z}, m \neq m_1, 1-m_1$, where $b \in F, b \neq 0.$

Proof. First, if there exists only one $m_1 \in Z$, $m_1 \neq 0, 1$ such that $f(m_1) \neq 0$ and $f(m) = 0$ for all $m \in Z$ and $m \neq m_1$. By Lemma 3.3 and a direct computation, R is a homogeneous Rota-Baxter operator. Thanks to Lemma 2.3, we can suppose $f(m_1) = 1$.

Second, if there exist only two distinct integers m_1, m_2 satisfying $m_1 + m_2 = 1$ and $m_1 \neq 0, 1$ such that $f(m_1) \neq 0$, $f(m_2) \neq 0$ and $f(m) = 0$ for all $m \in Z$ and $m \neq m_1, m \neq m_2$. Then for all $m \in Z$, we have $D(m_1, m_2, m) \neq 0$. By a direct computation, for all $l, m, n \in Z$, we have that $f(l)$, $f(m)$ and $f(n)$ satisfy Eq. (11) and Eq. (12). Therefore, R is a homogeneous Rota-Baxter operator. By Lemma 2.3, we can suppose $f(m_1) = 1$, $f(m_2) = f(1 - m_1) = b$, where $b \in F$ and $b \neq 0$.

Third, if R is a homogeneous Rota-Baxter operator satisfying that there exist two distinct integers m_1, m_2 such that $f(m_i) \neq 0$ and $f(m) = 0$, for all $m \in Z$ and $m \neq m_i$, $i = 1, 2$, where $m_1, m_2 \neq 0, 1$. Then there exists $m \in Z$ such that $D(m_1, m_2, m) \neq 0$. Thanks to Lemma 3.3,

$$f(m_1 + m_2 + m - 1) = 0.$$

Then $m_1 + m_2 + m - 1 \neq m_1$ and $m_1 + m_2 + m - 1 \neq m_2$, that is, $m \neq 1 - m_1$ and $m \neq 1 - m_2$. It follows that $1 - m_1 = m_2$.

Lastly, if R is a homogeneous Rota-Baxter operator satisfying $f(m_i) \neq 0$, and $f(m) = 0$ for all $m \neq m_i$, $1 \leq i \leq s$, $s \geq 3$. Then for every $1 \leq i \leq s$, $f(1 - m_i) \neq 0$.

In fact, if $f(1 - m_1) = 0$. From $D(m_1, m_2, 1 - m_1) \neq 0$, Eq. (11) and Eq. (12), we have $f(m_1 + m_2 + (1 - m_1) - 1) = f(m_2) = 0$. Contradiction. Therefore, $f(1 - m_1) \neq 0$. From $s \geq 3$, and similar discussions, we have that $f(1 - m_i) \neq 0$, for $1 \leq i \leq s$.

Therefore, s is even and $s \geq 4$ and we can suppose $m_1 < \dots < m_i < m_{i+1} < \dots < m_s$. Then there exists $m \in Z$, $m \neq 0, 1$ such that $f(m) = 0$ and $D(m_1, m_2, m) \neq 0$. Thanks to Eq. (11) and Eq. (12), $f(m_1 + m_2 + m - 1) = 0$. Then $m_1 + m_2 + m - 1 \neq m_s$, that is, $m \neq m_s - m_1 - m_2 + 1$. By the above discussion and $s \geq 4$, there exists $i \geq 3$ such that $m_s - m_1 - m_2 + 1 = 1 - m_i$. We obtain that $m_1 + m_2 = m_i + m_s$. Contradiction.

Summarizing above discussion, we obtain the result. \square

Lemma 3.16. *Let $R : A_\omega \rightarrow A_\omega$ be a homogeneous Rota-Baxter operator with $f(0) = f(1) = 0$, and satisfy that there exist infinite $m \in Z$ such that $f(m) \neq 0$. Then there exist infinite $n \in Z$ such that $f(n) = 0$, and for all $m \in Z$, if $f(m) \neq 0$, then $f(1 - m) \neq 0$, and*

$$f(m) + f(1 - m) = 0.$$

Proof. If there exists $m \in Z$ such that $f(m) \neq 0$, but $f(1 - m) = 0$. Then for all $n \in Z$ and $n \neq m, 1 - m$, by Eq. (11), Eq. (12) and $D(m, n, 1 - m) \neq 0$, we have $f(m + n + 1 - m - 1) = f(n) = 0$. Contradiction. Therefore, if $f(m) \neq 0$, then $f(1 - m) \neq 0$. Thanks to the result 8) in Corollary 3.14, there exist infinite $n \in Z$ such that $f(n) = 0$.

Now for distinct $2m, 2n \in Z$, $f(2m) \neq 0$ and $f(2n) \neq 0$ and $m \neq n$, by Eq. (12),

$$\begin{aligned} & f(1 - 2m)f(2m)f(2n) \\ &= ((f(1 - 2m)f(2m) + f(1 - 2m)f(2n) + f(2m)f(2n))f(1 - 2m + 2n + 2m)). \end{aligned}$$

It follows $f(1 - 2m) + f(2m) = 0$ for all $m \in Z$. The proof is completed. \square

Theorem 3.17. *Let $R : A_\omega \rightarrow A_\omega$ be a homogeneous Rota-Baxter operator with $f(0) = f(1) = 0$, and there exist infinite $m \in Z$ such that $f(m) \neq 0$. Then there exist positive integer m_0*

and s_0 satisfying $1 \leq s_0 < m_0$ such that $f(m) \neq 0$ if and only if $m \in W = \{2m_0k + 2s_0 \mid k \in \mathbb{Z}\} \cup \{1 - 2m_0k - 2s_0 \mid k \in \mathbb{Z}\}$.

Proof. By Lemma 3.16, we can suppose that $W = \{2x_k \mid k \in \mathbb{Z}\} \cup \{1 - 2x_k \mid k \in \mathbb{Z}\}$ is set of integers satisfying that $f(m) \neq 0$ if and only if $m \in W$. By Lemma 3.16 and Corollary 3.4, we can suppose that for $2x_k, 2x_s \in W$, $2x_k < 2x_s$ if and only if $k < s$ and $2x_{-1} < 0, 2x_0 > 0$.

Denote $x_1 - x_0 = m_0, x_2 - x_1 = m_1$. Then $m_0 > 0, m_1 > 0$. From $2x_0 \in W, -2x_1 + 1 \in W, 2x_2 \in W$ and the result 6) in Corollary 3.14, we have

$$2(x_2 - x_1 + x_0) = 2(m_0 + x_0 - x_0 + x_0 - m_1) = 2(x_2 - m_0) \in W.$$

Since $x_0 = x_1 - m_0 < x_2 - m_0 < x_2$, $x_2 - m_0 = x_1$, that is, $m_1 = m_0$.

Now suppose $x_k - x_{k-1} = m_0$ for $k > 0$. Denote $x_{k+1} - x_k = m_k$. According to the result 6) of Corollary 3.14, we have

$$2(x_{k+1} - x_k + x_{k-1}) = 2(m_k + x_k - x_k + x_k - m_0) = 2(x_{k+1} - m_0) \in W.$$

Thanks to $x_{k-1} = x_k - m_0 < x_{k+1} - m_0 < x_{k+1}$, $x_{k+1} - m_0 = x_k$, that is, $m_k = m_0$. Therefore, $2x_k = 2km_0 + 2x_0$, $k > 0, k \in \mathbb{Z}$.

Similar discussion we have $2x_k = 2km_0 + 2x_0$, for all $k < 0, k \in \mathbb{Z}$.

Therefore, $W = \{2km_0 + 2x_0 \mid k \in \mathbb{Z}, x_0 > 0\}$, where $m_0 > 0$.

By Lemma 3.16 and the result 1) in Corollary 3.14, $2x_1 + 2x_{-1} = 2x_0 + 2x_0 \notin W$, that is, m_0 is not a factor of x_0 . So there exist integers s_0 and q such that $1 \leq s_0 < m_0$ and $x_0 = qm_0 + s_0$.

Therefore, $2x_k = 2(k + q)m_0 + 2s_0$, for all $k \in \mathbb{Z}$. It follows the result. \square

For positive integer m_0 and s_0 with $1 \leq s_0 < m_0$, denote

$$W_{m_0, s_0} = \{2m_0k + 2s_0 \mid k \in \mathbb{Z}\} \cup \{1 - 2m_0k - 2s_0 \mid k \in \mathbb{Z}\}.$$

If f satisfies that $f(m) \neq 0$ if and only if $m \in W_{m_0, s_0}$, then W_{m_0, s_0} is called an (m_0, s_0) -**supporter** of R . By Lemma 2.3, we can suppose that $f(2s_0) = 1$.

Lemma 3.18. *Let $R : A_\omega \rightarrow A_\omega$ be a Homogeneous Rota-Baxter operator with (m_0, s_0) -supporter W_{m_0, s_0} , and $f(0) = f(1) = 0$. Then for all $k_i \in \mathbb{Z}$, and $k_i \neq k_j$, for $1 \leq i \neq j \leq 3$, we have*

$$(17) \quad \frac{1}{f(2m_0k_1 + 2s_0)} + \frac{1}{f(2m_0k_2 + 2s_0)} + \frac{1}{f(2m_0k_3 + 2s_0)} = \frac{1}{f(2m_0(k_1 + k_2 - k_3) + 2s_0)} \\ + \frac{1}{f(2m_0(k_1 - k_2 + k_3) + 2s_0)} + \frac{1}{f(2m_0(-k_1 + k_2 + k_3) + 2s_0)}.$$

Therefore,

$$(18) \quad \frac{1}{f(2m_0k + 2s_0)} + \frac{1}{f(-2m_0k + 2s_0)} = 2,$$

and $f(2m_0k + 2s_0) \neq \frac{1}{2}$ for all $k \in \mathbb{Z}$.

Proof By Lemma 3.3 and Lemma 3.16, for all $k_1, k_2, k_3 \in \mathbb{Z}$ and $k_1 \neq k_2$, we have

$$f(2m_0k_1 + 2s_0)f(2m_0k_2 + 2s_0)f(2m_0k_3 + 2s_0) \\ = (-f(2m_0k_1 + 2s_0)f(2m_0k_2 + 2s_0) + f(2m_0k_1 + 2s_0)f(2m_0k_3 + 2s_0) \\ + f(2m_0k_2 + 2s_0)f(2m_0k_3 + 2s_0))f(2m_0(k_1 + k_2 - k_3) + 2s_0) \neq 0.$$

Therefore,

$$\frac{1}{f(2k_1m_0 + 2s_0)} + \frac{1}{f(2k_2m_0 + 2s_0)} - \frac{1}{f(2k_3m_0 + 2s_0)} = \frac{1}{f(2(k_1 + k_2 - k_3)m_0 + 2s_0)}.$$

For the case $k_1 = -k_2$ and $k_3 = 0$, we obtain Eq.(18).

Similarly, for $k_1 \neq k_3$, we have

$$\frac{1}{f(2k_1m_0 + 2s_0)} + \frac{1}{f(2k_3m_0 + 2s_0)} - \frac{1}{f(2k_2m_0 + 2s_0)} = \frac{1}{f(2(k_1 + k_3 - k_2)m_0 + 2s_0)},$$

and for $k_2 \neq k_3$, we have

$$\frac{1}{f(2k_2m_0 + 2s_0)} + \frac{1}{f(2k_3m_0 + 2s_0)} - \frac{1}{f(2k_1m_0 + 2s_0)} = \frac{1}{f(2(k_2 + k_3 - k_1)m_0 + 2s_0)},$$

It follows Eq. (17). The result follows.

Theorem 3.19. *Let $R : A_\omega \rightarrow A_\omega$ be a linear map defined as Eq. (9) which satisfies that there exist infinite $m \in \mathbb{Z}$ such that $f(m) = f(0) = f(1) = 0$. Then R is a homogeneous Rota-Baxter operator on A_ω if and only if there exist positive integer m_0 and s_0 , and $a \in F$, such that W_{m_0, s_0} is an (m_0, s_0) -supporter of R , and*

$$(19) \quad f(2m_0k + 2s_0) = -f(1 - 2m_0k - 2s_0) = \frac{1}{ka - (k - 1)}, \quad \forall k \in \mathbb{Z},$$

where $1 \leq s_0 < m_0$ and $a \neq \frac{k-1}{k}$, for all $k \in \mathbb{Z}$ and $k \neq 0$.

Proof. The proof is completely similar to Theorem 3.12. □

Let $m_0 = 7$, $a = 2$, and $s_0 = 2$. By Theorem 3.19, the linear map $R : A_\omega \rightarrow A_\omega$ defined by for all $k \in \mathbb{Z}$,

$$R(L_{14k+4}) = \frac{1}{2k - (k - 1)}L_{14k+4} = \frac{1}{k + 1}L_{14k+4}, \quad R(L_{-14k-3}) = -\frac{1}{k + 1}L_{-14k-3},$$

and others are zero, is a Homogeneous Rota-Baxter operator of weight 0 with $(7, 2)$ -supporter

$$W_{7,2} = \{14k + 4 \mid k \in \mathbb{Z}\} \cup \{-14k - 3 \mid k \in \mathbb{Z}\}.$$

If $m_0 = 4$, $s_0 = 3$ and $a = \frac{3}{5}$, then the linear map $R : A_\omega \rightarrow A_\omega$ defined by for all $k \in \mathbb{Z}$,

$$R(L_{8k+6}) = \frac{5}{5 - 2k}L_{8k+6}, \quad R(L_{-8k-5}) = -\frac{5}{5 - 2k}L_{-8k-5},$$

and others are zero, is a homogeneous Rota-Baxter operator of weight 0 with the $(4, 3)$ -supporter

$$W_{4,3} = \{8k + 6 \mid k \in \mathbb{Z}\} \cup \{-8k - 5 \mid k \in \mathbb{Z}\}.$$

3.3. 3-Lie algebras constructed by A_ω and homogeneous Rota-Baxter operators. In the study of 3-Lie algebras, we know that construction of 3-Lie algebras from known algebras is always interesting. So in this section, we construct 3-Lie algebras from the 3-Lie algebra A_ω and the homogeneous Rota-Baxter operators.

Let $(A, [, ,])_{\lambda}$ be a 3-Lie algebra and R be a Rota-Baxter with weight λ . Using the notation in Eq. (5), we define a ternary operation $[, ,]_R$ on A by

$$(20) \quad [x_1, x_2, x_3]_R = \sum_{\emptyset \neq I \subseteq [3]} \lambda^{|I|-1} [\widehat{R}_I(R(x_1)), \widehat{R}_I(R(x_2)), \widehat{R}_I(R(x_3))], \quad \forall x, y, z \in A.$$

Therefore, in the case $\lambda = 0$, we have

$$(21) \quad [x_1, x_2, x_3]_R = [R(x), R(y), z] + [R(x), y, R(z)] + [x, R(y), R(z)], \quad \forall x, y, z \in A.$$

Theorem 3.20. [39] *Let $(A, [, ,])_{\lambda}$ be a 3-Lie algebra and R be a Rota-Baxter of weight λ . Then $(A, [, ,]_R)$ is 3-Lie algebra in the multiplication defined as Eq. (20), and R is also a Rota-Baxter operator of it.*

So if R is a homogeneous Rota-Baxter operator of the 3-Lie algebra A_ω of weight 0, then $(A, [, ,]_R)$ is a 3-Lie algebra in the multiplication defined as Eq. (21), where $A = A_\omega$ as vector spaces, and R is also a homogeneous Rota-Baxter operator of $(A, [, ,]_R)$.

Theorem 3.21. *Let $R : A_\omega \rightarrow A_\omega$ be a linear map defined as Eq. (9), then R is a homogeneous Rota-Baxter operator of weight 0 on 3-Lie algebra A_ω if and only if R is the one of the following*

$$R_{0_1}(L_0) = L_0, R_{0_1}(L_1) = bL_1, \text{ and } R_{0_1}(L_m) = 0, \text{ for all } m \in \mathbb{Z}, m \neq 0, 1.$$

$$R_{0_2}(L_m) = \begin{cases} L_0, & m = 0, \\ -L_1, & m = 1, \\ \frac{1}{ka-(k-1)}L_{2m_0k}, & m = 2m_0k \in W_{m_0}, \\ -\frac{1}{ka-(k-1)}L_{1-2m_0k}, & m = 1 - 2m_0k \in W_{m_0}, \\ 0, & \text{others.} \end{cases}$$

$$R_{0_3}(L_m) = \begin{cases} \frac{1}{ka-(k-1)}L_{2m_0k+2s_0}, & m = 2m_0k + 2s_0 \in W_{m'_0, s_0}, \\ -\frac{1}{ka-(k-1)}L_{1-2m_0k-2s_0}, & m = 1 - 2m_0k - 2s_0 \in W_{m'_0, s_0}, \\ 0, & \forall m \notin W_{m'_0, s_0}. \end{cases}$$

$$R_{0_4}(L_m) = \begin{cases} L_{m_1}, & m = m_1, \\ 0, & m \neq m_1. \end{cases}$$

$$R_{0_5}(L_m) = \begin{cases} L_{m_1}, & m = m_1, \\ bL_{1-m_1}, & m = 1 - m_1, \\ 0, & m \neq m_1, 1 - m_1. \end{cases}$$

where $m_1, m_0, m'_0, s_0 \in \mathbb{Z}$, $m_1 \neq 0, 1$; $m_0 > 0$; $1 \leq s_0 < m'_0$; $a, b, c \in F$, $a \neq \frac{k-1}{k}$, $b \neq 0$, $W_{m_0} = \{2m_0k \mid k \in \mathbb{Z}\} \cup \{1 - 2m_0k \mid k \in \mathbb{Z}\}$, $W_{m'_0, s_0} = \{2m'_0k + 2s_0 \mid k \in \mathbb{Z}\} \cup \{1 - 2m'_0k - 2s_0 \mid k \in \mathbb{Z}\}$.

Proof. The result follows from Theorem 3.2, Theorem 3.12, Theorem 3.15 and Theorem 3.19. \square

For convenience, denote $\lambda_k = ka - (k - 1)$, for all $k \in \mathbb{Z}$, $k \neq \frac{k-1}{k}$, and the multiplication $[, ,]_{R_{0_i}}$ defined as Eq. (21) by $[, ,]_i$, $1 \leq i \leq 5$. Then we obtain 3-Lie algebras $(A, [, ,]_i)$ with the

homogeneous Rota-Baxter operators R_{0_i} for $1 \leq i \leq 5$, where $A = A_\omega$ as vector spaces. And we omit the zero product of basis vectors in the multiplication of 3-Lie algebras $[A, [, ,]_i]$, for $1 \leq i \leq 5$.

1) $([A, [, ,]_1])$ with the multiplication

$$[L_0, L_1, L_m]_1 = c(2m - 1 + (-1)^m)L_m, \text{ for all } m \in Z, m \neq 0, 1, b \in F, b \neq 0.$$

2) $([A, [, ,]_2])$ with the multiplication

$$[L_0, L_1, L_{2m}]_2 = -4mL_{2m},$$

$$[L_0, L_1, L_{2m+1}]_2 = -4mL_{2m+1},$$

$$[L_0, L_{1-2m_0k_1}, L_{2m}]_2 = \frac{-4m}{\lambda_{k_1}}L_{2m-2m_0k_1},$$

$$[L_0, L_{2m_0k_1}, L_{2m+1}]_2 = -\frac{4m_0k_1}{\lambda_{k_1}}L_{2m+2m_0k_1},$$

$$[L_1, L_{2m_0k_1}, L_{2m}]_2 = -\frac{4m_0k_1-4m}{\lambda_{k_1}}L_{2m+2m_0k_1},$$

$$[L_1, L_{2m_0k_1}, L_{2m+1}]_2 = \frac{4m}{\lambda_{k_1}}L_{2m+2m_0k_1+1},$$

$$[L_1, L_{1-2m_0k_1}, L_{2m}]_2 = -\frac{4m_0k_1}{\lambda_{k_1}}L_{2m-2m_0k_1+1},$$

$$[L_0, L_{2m_0k_1}, L_{1-2m_0k_2}]_2 = \frac{-4m_0k_1(\lambda_{k_2}-\lambda_{k_1}-1)}{\lambda_{k_1}\lambda_{k_2}}L_{2m_0(k_1-k_2)},$$

$$[L_0, L_{1-2m_0k_1}, L_{1-2m_0k_2}]_2 = \frac{4m_0(k_1-k_2)(-\lambda_{k_2}-\lambda_{k_1}+1)}{\lambda_{k_1}\lambda_{k_2}}L_{-2m_0(k_1+k_2)+1},$$

$$[L_0, L_{1-2m_0k_1}, L_{2m+1}]_2 = -\frac{4m+4m_0k_1}{\lambda_{k_1}}L_{2m-2m_0k_1+1},$$

$$[L_1, L_{2m_0k_1}, L_{1-2m_0k_2}]_2 = \frac{4m_0k_2(\lambda_{k_1}-\lambda_{k_2}-1)}{\lambda_{k_1}\lambda_{k_2}}L_{2m_0(k_1-k_2)+1},$$

$$[L_1, L_{2m_0k_1}, L_{2m_0k_2}]_2 = \frac{4m_0(k_1-k_2)(-\lambda_{k_2}-\lambda_{k_1}+1)}{\lambda_{k_1}\lambda_{k_2}}L_{2m_0(k_1+k_2)},$$

$$[L_{2m_0k_1}, L_{1-2m_0k_2}, L_{2m}]_2 = -\frac{4m-4m_0k_1}{\lambda_{k_1}\lambda_{k_2}}L_{2m+2m_0(k_1-k_2)},$$

$$[L_{2m_0k_1}, L_{1-2m_0k_2}, L_{2m+1}]_2 = -\frac{4m+4m_0k_2}{\lambda_{k_1}\lambda_{k_2}}L_{2m+2m_0(k_1-k_2)+1},$$

$$[L_{2m_0k_1}, L_{2m_0k_2}, L_{1-2m_0k_3}]_2 = \frac{4m_0(k_1-k_2)(\lambda_{k_3}-\lambda_{k_2}-\lambda_{k_1})}{\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}}L_{2m_0(k_1+k_2-k_3)},$$

$$[L_{2m_0k_1}, L_{1-2m_0k_2}, L_{1-2m_0k_3}]_2 = \frac{4m_0(k_2-k_3)(\lambda_{k_1}-\lambda_{k_2}-\lambda_{k_3})}{\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}}L_{2m_0(k_1-k_2-k_3)+1},$$

$$[L_{2m_0k_1}, L_{2m_0k_2}, L_{2m+1}]_2 = \frac{4m_0(k_1-k_2)}{\lambda_{k_1}\lambda_{k_2}}L_{2m+2m_0(k_1+k_2)},$$

$$[L_{1-2m_0k_1}, L_{1-2m_0k_2}, L_{2m}]_2 = \frac{4m_0(k_1-k_2)}{\lambda_{k_1}\lambda_{k_2}}L_{2m-2m_0(k_1+k_2)+1},$$

for all $2m+1, 2m \in Z$ and $2m, 2m+1 \notin W_{m_0}$, where $m_0 \in Z, m_0 > 0$,

2) $([A, [, ,]_3])$ with the multiplication

$$[L_{2m_0k_1+2s_0}, L_{2m_0k_2+2s_0}, L_{1-2m_0k_3-2s_0}]_3 = \frac{4m_0(k_1-k_2)(\lambda_{k_3}-\lambda_{k_2}-\lambda_{k_1})}{\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}}L_{2m_0(k_1+k_2-k_3)+2s_0},$$

$$[L_{2m_0k_1+2s_0}, L_{1-2m_0k_2-2s_0}, L_{1-2m_0k_3-2s_0}]_3 = \frac{4m_0(k_2-k_3)(\lambda_{k_1}-\lambda_{k_2}-\lambda_{k_3})}{\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}}L_{2m_0(k_1-k_2-k_3)-2s_0+1},$$

$$\begin{aligned}
[L_{2m_0k_1+2s_0}, L_{1-2m_0k_2-2s_0}, L_{2m+1}]_3 &= -\frac{4(m+m_0k_2+s_0)}{\lambda_{k_1}\lambda_{k_2}}L_{2m+2m_0(k_1-k_2)+1}, \\
[L_{2m_0k_1+2s_0}, L_{1-2m_0k_2-2s_0}, L_{2m}]_3 &= \frac{4(m-m_0k_1-s_0)}{\lambda_{k_1}\lambda_{k_2}}L_{2m+2m_0(k_1-k_2)}, \\
[L_{2m_0k_1+2s_0}, L_{2m_0k_2+2s_0}, L_{2m+1}]_3 &= \frac{4m_0(k_1-k_2)}{\lambda_{k_1}\lambda_{k_2}}L_{2m+2m_0(k_1+k_2)+4s_0}, \\
[L_{1-2m_0k_1-2s_0}, L_{1-2m_0k_2-2s_0}, L_{2m}]_3 &= \frac{4m_0(k_1-k_2)}{\lambda_{k_1}\lambda_{k_2}}L_{2m-2m_0(k_1+k_2)-4s_0+1},
\end{aligned}$$

for all $2m+1, 2m \in Z$ and $2m, 2m+1 \notin W_{m_0, s_0}$, where $m_0, s_0 \in Z, 1 \leq s_0 < m_0$.

4) $([A, [,],]_4)$ is an abelian algebras.

6) $([A, [,],]_5)$ with the multiplication

$$[L_{m_1}, L_{1-m_1}, L_m]_5 = bD(m_1, 1-m_1, m)L_m, \text{ for all } m \in Z, m \neq m_1, \text{ where } m_1 \in Z, m_1 \neq 0, 1, b \in F, b \neq 0.$$

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